

## Local SU(2) Symmetry: The SU(2) Covariant Derivative and the SU(2) Gauge Fields

Local SU(2) gauge transformations have the form

$$\Psi(x) \rightarrow \Psi'(x) = e^{i\frac{g}{2}\vec{\tau} \cdot \vec{\alpha}(x)} \Psi(x)$$

where the factor  $g$  is inserted to represent the coupling strength.

Just as in the case of electromagnetic interactions no Lagrangian for a free particle can be Lorentz invariant under this local gauge transformation. To make it Lorentz invariant, the derivative must be replaced by a covariant derivative. This way,  $D^\mu \Psi$  transforms the same way  $\Psi$  does, whereas  $\partial^\mu \Psi$  does not.

In the SU(2) case

$$\partial^\mu \Psi'(x) = e^{i\frac{g}{2}\vec{\tau} \cdot \vec{\alpha}(x)} \partial^\mu \Psi(x) + i\frac{g}{2}\vec{\tau} \cdot \partial^\mu \vec{\alpha}(x) e^{i\frac{g}{2}\vec{\tau} \cdot \vec{\alpha}(x)} \Psi(x)$$

where it is the 2nd term that breaks the covariance.

The covariant derivative  $D^\mu$  must act like

$$D^\mu \Psi'(x) = e^{i\frac{g}{2}\vec{\tau} \cdot \vec{\alpha}(x)} D^\mu \Psi(x)$$

The form of  $D^\mu$  is just postulated, then justified by the fact that it works.

$$D^\mu \equiv \partial^\mu + i\frac{g}{2}\vec{\tau} \cdot \vec{W}^\mu$$

$$\vec{W}^\mu = (W_1^\mu, W_2^\mu, W_3^\mu)$$

The  $\vec{W}^\mu$  are the SU(2) gauge fields, analogous to the U(1) gauge field  $A^\mu$ .

$$\begin{aligned}\vec{\epsilon} \cdot \vec{W}^\mu &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} W_1^\mu + \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} W_2^\mu + \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} W_3^\mu \\ &= \begin{bmatrix} 0 & W_1^\mu \\ W_1^\mu & 0 \end{bmatrix} + \begin{bmatrix} 0 & -iW_2^\mu \\ iW_2^\mu & 0 \end{bmatrix} + \begin{bmatrix} W_3^\mu & 0 \\ 0 & -W_3^\mu \end{bmatrix} \\ &= \begin{bmatrix} W_3^\mu & W_1^\mu - iW_2^\mu \\ W_1^\mu + iW_2^\mu & -W_3^\mu \end{bmatrix}\end{aligned}$$

Remember that the three gauge fields  $\vec{W}^\mu$  are spacetime dependent.

Let's examine the SU(2) transformation in infinitesimal form.

$$\psi' = (1 + \frac{iq}{2} \vec{\epsilon} \cdot \vec{\epsilon}(x)) \psi$$

$$\partial^\mu \psi' = (1 + \frac{iq}{2} \vec{\epsilon} \cdot \vec{\epsilon}(x)) \partial^\mu \psi + \frac{iq}{2} \vec{\epsilon} \cdot (\partial^\mu \vec{\epsilon}) \psi$$

We again see the noncovariant term. Let's use the covariant derivative instead.

$$D^\mu \psi' = (1 + \frac{iq}{2} \vec{\epsilon} \cdot \vec{\epsilon}(x)) D^\mu \psi$$

$$(\partial^\mu + \frac{iq}{2} \vec{\epsilon} \cdot \vec{W}'^\mu) [1 + \frac{iq}{2} \vec{\epsilon} \cdot \vec{\epsilon}(x)] \psi = [1 + \frac{iq}{2} \vec{\epsilon} \cdot \vec{\epsilon}(x)] (\partial^\mu + \frac{iq}{2} \vec{\epsilon} \cdot \vec{W}^\mu) \psi$$

So far, we do not know how the gauge fields  $W^\mu$  transform (notice that both  $\vec{W}'^\mu$  and  $\vec{W}^\mu$  appear). We proceed by assuming that the previous equality does, in fact, hold; and determine the transformation law for  $\vec{W}^\mu$  from it.

The previous equality involves an infinitesimal transformation  
so the transformation of  $\vec{W}^\mu$  must look something like

$$\vec{W}^\mu \rightarrow \vec{W}'^\mu = \vec{W}^\mu + \delta \vec{W}^\mu$$

$$\Rightarrow \left[ J^\mu + \frac{i\bar{q}}{2} \vec{\tau} \cdot (\vec{W}^\mu + \delta \vec{W}^\mu) \right] \left[ 1 + \frac{i\bar{q}}{2} \vec{\tau} \cdot \vec{E}(\mu) \right] \Psi = \left[ 1 + \frac{i\bar{q}}{2} \vec{\tau} \cdot \vec{E}(\mu) \right] \left[ J^\mu + \frac{i\bar{q}}{2} \vec{\tau} \cdot \vec{W}'^\mu \right] \Psi$$

$$\Rightarrow \left[ J^\mu + \frac{i\bar{q}}{2} \vec{\tau} \cdot \vec{W}'^\mu + \frac{i\bar{q}}{2} \vec{\tau} \cdot \delta \vec{W}^\mu \right] \left[ 1 + \frac{i\bar{q}}{2} \vec{\tau} \cdot \vec{E}(\mu) \right] \Psi = \left[ 1 + \frac{i\bar{q}}{2} \vec{\tau} \cdot \vec{E}(\mu) \right] \left[ J^\mu + \frac{i\bar{q}}{2} \vec{\tau} \cdot \vec{W}'^\mu \right] \Psi$$

$$\Rightarrow \left[ J^\mu + \frac{i\bar{q}}{2} \vec{\tau} \cdot (\vec{\tau} \cdot \vec{E}) + \frac{i\bar{q}}{2} \vec{\tau} \cdot \vec{W}'^\mu - \frac{1}{4} \bar{q}^2 (\vec{\tau} \cdot \vec{W}'^\mu) (\vec{\tau} \cdot \vec{E}) + \frac{i\bar{q}}{2} \vec{\tau} \cdot \delta \vec{W}^\mu - \frac{1}{4} \bar{q}^2 (\vec{\tau} \cdot \delta \vec{W}^\mu) (\vec{\tau} \cdot \vec{E}) \right] \Psi$$

$$= \left[ J^\mu + \frac{i\bar{q}}{2} \vec{\tau} \cdot \vec{W}'^\mu + \frac{i\bar{q}}{2} \vec{\tau} \cdot \vec{E} J^\mu - \frac{1}{4} \bar{q}^2 (\vec{\tau} \cdot \vec{E}) (\vec{\tau} \cdot \vec{W}'^\mu) \right] \Psi$$

$$\Rightarrow \left[ \frac{i\bar{q}}{2} \vec{\tau} \cdot \vec{\tau} \cdot \vec{E} - \frac{1}{4} \bar{q}^2 (\vec{\tau} \cdot \vec{W}'^\mu) (\vec{\tau} \cdot \vec{E}) + \frac{i\bar{q}}{2} \vec{\tau} \cdot \delta \vec{W}^\mu \right] \Psi$$

$$= \left[ \frac{i\bar{q}}{2} \vec{\tau} \cdot \vec{E} J^\mu - \frac{1}{4} \bar{q}^2 (\vec{\tau} \cdot \vec{E}) (\vec{\tau} \cdot \vec{W}'^\mu) \right] \Psi$$

$$\Rightarrow \frac{i\bar{q}}{2} \vec{\tau} \cdot J^\mu (\vec{E} \Psi) - \frac{1}{4} \bar{q}^2 (\vec{\tau} \cdot \vec{W}'^\mu) (\vec{\tau} \cdot \vec{E}) \Psi + \frac{i\bar{q}}{2} \vec{\tau} \cdot \delta \vec{W}^\mu \Psi$$

$$= \frac{i\bar{q}}{2} \vec{\tau} \cdot (\vec{E} J^\mu \Psi) - \frac{1}{4} \bar{q}^2 (\vec{\tau} \cdot \vec{E}) (\vec{\tau} \cdot \vec{W}'^\mu) \Psi$$

$$\Rightarrow \frac{i\bar{q}}{2} \vec{\tau} \cdot \delta \vec{W}^\mu \Psi = \frac{i\bar{q}}{2} \vec{\tau} \cdot \left[ \vec{E} (J^\mu \Psi) - J^\mu (\vec{E} \Psi) \right] + \frac{1}{4} \bar{q}^2 \left[ (\vec{\tau} \cdot \vec{W}'^\mu) (\vec{\tau} \cdot \vec{E}) - (\vec{\tau} \cdot \vec{E}) (\vec{\tau} \cdot \vec{W}'^\mu) \right] \Psi$$

$$\Rightarrow \frac{i\bar{q}}{2} \vec{\tau} \cdot \delta \vec{W}^\mu \Psi = \frac{i}{2} \bar{q} \vec{\tau} \cdot [-(J^\mu \vec{E}) \Psi] + \frac{1}{4} \bar{q}^2 \left[ (\vec{\tau} \cdot \vec{W}'^\mu) (\vec{\tau} \cdot \vec{E}) - (\vec{\tau} \cdot \vec{E}) (\vec{\tau} \cdot \vec{W}'^\mu) \right] \Psi$$

$$\Rightarrow i\bar{q} \frac{\vec{\tau} \cdot \delta \vec{W}^\mu}{2} = -i\bar{q} \frac{\vec{\tau} \cdot J^\mu \vec{E}(\mu)}{2} + (i\bar{q})^2 \left[ \left( \frac{\vec{\tau} \cdot \vec{E}}{2} \right) \left( \frac{\vec{\tau} \cdot \vec{W}'^\mu}{2} \right) - \left( \frac{\vec{\tau} \cdot \vec{W}'^\mu}{2} \right) \left( \frac{\vec{\tau} \cdot \vec{E}}{2} \right) \right]$$

$$\Rightarrow \vec{\tau} \cdot \delta \vec{W}^\mu = -\vec{\tau} \cdot \partial^\mu \vec{\epsilon} - g \underbrace{[(\vec{\epsilon} \cdot \vec{\epsilon})(\vec{\tau} \cdot \vec{W}^\mu) - (\vec{\epsilon} \cdot \vec{W}^\mu)(\vec{\tau} \cdot \vec{\epsilon})]}_{(\vec{\epsilon} \cdot \vec{W} + i\vec{\tau} \cdot \vec{\epsilon} \times \vec{W}) - (\vec{W} \cdot \vec{\epsilon} + i\vec{\tau} \cdot \vec{W} \times \vec{\epsilon})}$$

$$\underbrace{i\vec{\tau} \cdot (\vec{\epsilon} \times \vec{W} - \vec{W} \times \vec{\epsilon})}_{i\vec{\tau} \cdot (\vec{\epsilon} \times \vec{W} + \vec{\epsilon} \times \vec{W})}$$

$$2i\vec{\tau} \cdot (\vec{\epsilon} \times \vec{W})$$

$$\Rightarrow \vec{\tau} \cdot \delta \vec{W}^\mu = -\vec{\tau} \cdot \partial^\mu \vec{\epsilon} - g \vec{\tau} \cdot (\vec{\epsilon} \times \vec{W}^\mu)$$

$$\boxed{\Rightarrow \delta \vec{W}^\mu = -\partial^\mu \vec{\epsilon}(x) - g [\vec{\epsilon}(x) \times \vec{W}^\mu]}$$

Generalizing from global to local transformations introduces the extra  $-\partial^\mu \vec{\epsilon}(x)$  term. Hence, the gauge fields for a local SU(2) gauge (phase) transformation transform as

$$\boxed{\vec{W}'^\mu = \vec{W}^\mu - \partial^\mu \vec{\epsilon}(x) - g [\vec{\epsilon}(x) \times \vec{W}^\mu]}$$

Go on to do  
bottom of p.39 in  
"Collider Physics"  
Same as for  $U(1)$   
Case.

also starting w/  
14.6S of  
"Quarks & Leptons"